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Discrete Optimization

A modified LPT algorithm for the two uniform parallel machine makespan minimization problem

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Abstract

We propose a modified longest processing time (MLPT) heuristic algorithm for the two uniform machine makespan minimization problem. The MLPT algorithm schedules the three longest jobs optimally first, followed by the remaining jobs sequenced according to the LPT rule. We prove the tight worst-case ratio bound of $\sqrt{1.5} = 1.2247$ for the MLPT algorithm which is an improvement over the tight worst-case ratio bound of 1.28 for the LPT algorithm.

Keywords: Scheduling; Uniform parallel machines; LPT; Approximation algorithms

1. Introduction

One of the earliest scheduling rules for parallel machine scheduling problems is the longest processing time (LPT) priority rule in which the jobs are listed in a nonincreasing processing time order and the next job on the list is scheduled on the machine on which it will finish the earliest. Graham (1969) derived the tight worst-case ratio bound of the LPT rule for the identical parallel machine makespan minimization problem and also observed that the derived bound can be improved if the k longest jobs are scheduled optimally first, followed by the remaining jobs scheduled according to the LPT rule. Koulamas and Kyparisis (2008) demonstrated the robustness of this approach by implementing it to a two identical parallel machine scheduling problem with the objective of minimizing the sum of squares of the machine completion times.

The objective of this paper is to investigate whether or not the above approach extends to a uniform parallel machine environment; more precisely, whether or not it can be implemented to the two related (uniform) parallel machine makespan minimization ($Q2\|C_{\max}$) problem which is defined as follows. We assume that there are n jobs J_j with processing times p_j, $j = 1, \ldots, n$ to be scheduled nonpreemptively on two uniform parallel machines. Machine $M1$ (the "fast" machine) has speed 1 and machine $M2$ (the "slow" machine) has speed $1/q$, where $q \geqslant 1$. The effective processing time of job J_j is p_j on machine $M1$ and qp_j on machine $M2$. The completion time of job J_j is denoted as C_j, $j = 1, \ldots, n$. The machine completion time of machine Mi, $i = 1, 2$, defined as the completion time of the last job scheduled on it, is denoted as C_{Mi}, $i = 1, 2$. The maximum machine completion time (makespan) is defined as $C_{\max} = \max_{j=1,\ldots,n}\{C_j\} = \max\{C_{M1}, C_{M2}\}$. The objective is to minimize $C_{\max}$. Gonzalez et al. (1977) showed that the LPT rule yields the tight worst-case ratio bound of 1.28 for the $Q2\|C_{\max}$ problem. In this paper, we propose a modified LPT (MLPT) priority rule in which the three longest jobs are scheduled optimally first, followed by the remaining jobs scheduled according to the LPT rule. We show that the MLPT rule has the tight worst-case bound of 1.22, an improvement over the LPT bound of 1.28.

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We close this section by surveying the related literature. Mireault et al. (1997) determined the parametric LPT bound for the whole spectrum of q, the speed ratio of the two uniform machines. Epstein and Favrholdt (2005) proposed three variants of the LPT algorithm which improve the LPT bound for certain q values without improving the overall nonparametric LPT bound of 1.28. Additional references for the related on-line version of the $Q2\|C_{\max}$ problem can be found in Koulamas and Kyparisis (2006).

The rest of the paper is organized as follows. The proposed MLPT heuristic is presented in the next section followed by the derivation of its tight worst-case ratio bound in Section 3. An extension to the MLPT heuristic is presented in Section 4 and some concluding remarks are presented in Section 5.

2. The modified LPT algorithm

The proposed algorithm MLPT for the $Q2\|C_{\max}$ problem can be summarized as follows.

Algorithm MLPT

Step 1: Sort all jobs J_j, $j = 1, \ldots, n$, in the nonincreasing order of their p_j values, that is $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n$. Let S denote the resulting list (sequence); (without loss of generality, we assume that $S = \{1, 2, \ldots, n\}$ and we denote job J_j simply as job j).

Step 2: If $n \leqslant 2$, then schedule all jobs optimally by using the LPT rule.

If $n \geqslant 3$, then schedule the first (longest) 3 jobs in S optimally followed by the remaining jobs scheduled according to the LPT rule.

The running time of algorithm MLPT is $O(n \log n + c)$, where c represents the constant time needed to schedule the three longest jobs optimally.

Following Mireault et al. (1997), we introduce the following notation. Let $C^H(P,q)$ and $C^*(P,q)$ denote the makespan of the algorithm MLPT and the optimal makespan, respectively, for the problem instance (P,q). Let the relative error for MLPT with respect to problem instance (P,q) be defined as

$$\mathrm{REL}(P,q) = \frac{C^H(P,q) - C^*(P,q)}{C^*(P,q)}.$$

The parametric worst-case ratio bound for MLPT is defined as

$$R(q) = \sup\{\mathrm{REL}(P,q) : P\}$$

and the overall nonparametric worst-case ratio bound ρ is defined as

$$\rho = \sup\{R(q) : q \geqslant 1\}. \tag{1}$$

The derivation of ρ is detailed in the next section.

3. The worst-case ratio bound of MLPT

For a problem instance (P,q), let $T(1,j)$, $T(2,j)$ denote the sums of processing times p_j of all jobs assigned by MLPT to machines $M1$ and $M2$, respectively, after job j has been assigned. Thus, the total effective processing time for MLPT is $T(1,n)$ on $M1$ and $qT(2,n)$ on $M2$. We first adapt Lemma 2 in Mireault et al. (1997) (developed for the LPT heuristic) to the MLPT heuristic.

Proposition 1. *For any* $n \geqslant 1$,

$$C^H(P,q) - C^*(P,q) \leqslant \frac{qp_n}{q+1}. \tag{2}$$

Proof. Since $T(1,n-1)$, $T(2,n-1)$ denote the sums of processing times p_j of all jobs assigned by MLPT to machines $M1$ and $M2$, respectively, after job $n-1$ (and before job n) has been assigned, we can write

$$C^H(P,q) = \min\{T(1,n-1) + p_n, qT(2,n-1) + qp_n\}.$$

This implies that

$$C^H(P,q) \leqslant T(1,n-1) + p_n, \tag{3}$$

$$C^H(P,q) \leqslant qT(2,n-1) + qp_n. \tag{4}$$

Let $C^*(i)$, $i = 1, 2$, denote the sums of processing times p_j of all jobs assigned to machines $M1$ and $M2$, respectively, in the optimal schedule. Since $C^*(1) \leqslant C^*(P,q)$ and $qC^*(2) \leqslant C^*(P,q)$, □

$$T(1,n-1)+T(2,n-1)+p_n=C^*(1)+C^*(2)\leqslant[1+\frac{1}{q}]C^*(P,q). \tag{5}$$

The equality in (5) is a result of the observation that the total processing time of all jobs is the same for both the MLPT and the optimal solutions, respectively. By adding $q/(q+1)$ times inequality (3), $1/(q+1)$ times inequality (4) and $q/(q+1)$ times inequality (5), we obtain

$$\frac{q}{q+1}C^H(P,q)+\frac{1}{q+1}C^H(P,q)+\frac{q}{q+1}[T(1,n-1)+T(2,n-1)+p_n]$$
$$\leqslant\frac{q}{q+1}[T(1,n-1)+p_n]+\frac{1}{q+1}[qT(2,n-1)+qp_n]+\frac{q}{q+1}\left[1+\frac{1}{q}\right]C^*(P,q),$$

or, $C^H(P,q)\leqslant\frac{q}{q+1}p_n+C^*(P,q)$ which is equivalent to (2). □

The next result is an adaptation of Proposition 1 in Mireault et al. (1997) (developed for the LPT heuristic) to the MLPT heuristic.

Proposition 2. *For any* $n\geqslant 1$,

$$R(q)\leqslant\frac{1}{2q+1}. \tag{6}$$

Proof. If $n\leqslant 3$, the MLPT algorithm supplies an optimal solution, therefore $R(q)=0$ in this case. Assume that $n\geqslant 4$. If $T(2,n-1)=0$ then the MLPT algorithm assigns the first $n-1$ jobs to $M1$ in which case $C^H(P,q)=C^*(P,q)$ and $\mathrm{REL}(P,q)=0$. Thus, $C^H(P,q)\neq C^*(P,q)$ implies that $T(2,n-1)>0$, or that

$$p_n\leqslant T(2,n-1). \tag{7}$$

By adding $2q$ times inequality (3), $2q$ times inequality (5), q times inequality (7) and inequality (4), we obtain

$$2qC^H(P,q)+C^H(P,q)+2q[T(1,n-1)+T(2,n-1)+p_n]+qp_n$$
$$\leqslant 2q[T(1,n-1)+p_n]+[qT(2,n-1)+qp_n]+2q\left[1+\frac{1}{q}\right]C^*(P,q)+qT(2,n-1),$$

or, $(2q+1)C^H(P,q)\leqslant(2q+2)C^*(P,q)$, which is equivalent to (6). □

Our MLPT heuristic supplies the optimal solution when $n\leqslant 3$, therefore we should determine its worst-case ratio bound ρ for all remaining n values. For each of these remaining n values, we first derive $R(q)$ for all $q\geqslant 1$ values and then determine ρ according to (1).

3.1. The worst-case ratio bound of MLPT when $n\geqslant 6$

Let us denote by $n_{OPT}(i)$ the number of jobs assigned to Mi, $i=1,2$ in the optimal solution $C^*(P,q)$. The thrust of the proof is to first determine $R(q)$ for each of the five q ranges depicted in Fig. 1 and then trivially obtain ρ according to (1).

Lemma 1. *For* $n\geqslant 6$, $R(q)\leqslant\frac{1}{3q+3}$, $q\in[1,\frac{4}{3}]$ *and* $R(q)\leqslant\frac{q}{4q+4}$, $q\in[\frac{4}{3},2]$.

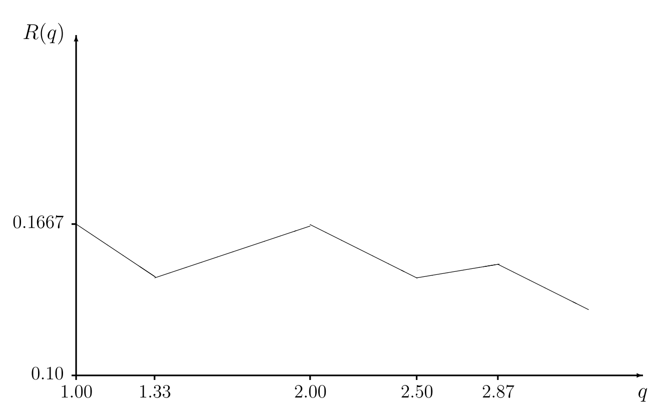

Fig. 1. Bound on $R(q)$ when $n\geqslant 6$.

Proof

1-1. $n_{\mathrm{OPT}}(1) \leqslant 3$; then $n_{\mathrm{OPT}}(2) \geqslant 3$ and $C^*(P,q) \geqslant 3qp_n$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{3qp_n} = \frac{1}{3q+3},\ q \geqslant 1.$$

The inequality $\frac{1}{3q+3} \leqslant \frac{q}{4q+4}$ when $q \in [\frac{4}{3}, 2]$ completes the proof of 1-1.

1-2. $n_{\mathrm{OPT}}(1) \geqslant 4$; then $C^*(P,q) \geqslant 4p_n$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{4p_n} = \frac{q}{4q+4},\ q \geqslant 1.$$

The inequality $\frac{q}{4q+4} \leqslant \frac{1}{3q+3}$ when $q \in [1, \frac{4}{3}]$ completes the proof of 1-2. □

Lemma 2. *For $n \geqslant 6$, $R(q) \leqslant \frac{1}{2q+2}$, $q \in [2, 2.5]$ and $R(q) \leqslant \frac{q}{5q+5}$, $q \in [2.5, 1+\sqrt{3.5}]$.*

Proof

2-1. $n_{\mathrm{OPT}}(2) \leqslant 1$; then $n_{\mathrm{OPT}}(1) \geqslant 5$ and $C^*(P,q) \geqslant 5p_n$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{5p_n} = \frac{q}{5q+5},\ q \geqslant 1.$$

The inequality $\frac{q}{5q+5} \leqslant \frac{1}{2q+2}$ when $q \in [2, 2.5]$ completes the proof of 2-1.

2-2. $n_{\mathrm{OPT}}(2) \geqslant 2$; then $C^*(P,q) \geqslant 2qp_n$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{2qp_n} = \frac{1}{2q+2}, q \geqslant 1. \quad (8)$$

The inequality $\frac{1}{2q+2} \leqslant \frac{q}{5q+5}$ when $q \in [2.5, 1+\sqrt{3.5}]$ completes the proof of 2-2. □

Lemmas 1 and 2 combined with Proposition 2 for $q \in [1+\sqrt{3.5}, \infty)$ lead to the following proposition which is stated next without proof (see also Fig. 1).

Proposition 3. *For $n \geqslant 6$, $\rho \leqslant \frac{1}{6} = 0.1667$.*

3.2. The worst-case ratio bound of MLPT when n = 4

Let us denote by $H(j_1,\ldots,j_{n_1}|k_1,\ldots,k_{n_2})$ and $O(j_1,\ldots,j_{n_1}|k_1,\ldots,k_{n_2})$ the solution provided by algorithm MLPT and the optimal solution, respectively, where jobs $j_1,\ldots,j_{n_1}$ are assigned to $M1$, jobs $k_1,\ldots,k_{n_2}$ are assigned to $M2$; the job sets $\{j_1,\ldots,j_{n_1}\}$ and $\{k_1,\ldots,k_{n_2}\}$ are mutually exclusive and $n_1+n_2=n$. For example, $H(2|1,3)$ denotes an MLPT solution with job 2 assigned to $M1$ and jobs 1, 3 assigned to $M2$ and $O(1,2,3|-)$ denotes an optimal solution with jobs 1, 2, 3 assigned to $M1$ and no job assigned to $M2$.

There are eight possible configurations for $C^*(P,q)$ when $n=3$, three of which are "dominated", that is they are no better than some other configuration. The five "nondominated" configurations are

$$O(1|2,3), O(1,2|3), O(1,3|2), O(2,3|1), O(1,2,3|-).$$

Since algorithm MLPT first finds an optimal schedule for the longest three jobs and then schedules all subsequent jobs using the LPT rule, there are ten possible configurations for $C^H(P,q)$ when $n=4$ (each $C^*(P,q)$ configuration for $n=3$ generates two possible $C^H(P,q)$ configurations for $n=4$ by assigning the fourth job in the LPT list to either machine $M1$ or $M2$). These ten configurations are

$$H(1,4|2,3), H(1|2,3,4), H(1,2,4|3), H(1,2|3,4), H(1,3,4|2), H(1,3|2,4), H(2,3,4|1), H(2,3|1,4), H(1,2,3,4|-), H(1,2,3|4).$$

Also, among the sixteen possible configurations for $C^*(P,q)$ when $n=4$, the following ten configurations are nondominated

$$O(1|2,3,4), O(1,2|3,4), O(1,3|2,4), O(1,4|2,3), O(2,3|1,4), O(1,2,3|4), O(1,2,4|3), O(1,3,4|2), O(2,3,4|1), O(1,2,3,4|-).$$

Consequently, we need to implicitly or explicitly consider one hundred possible matchings between the ten possible configurations for $C^H(P,q)$ and the ten nondominated configurations for $C^*(P,q)$ when $n=4$ in order to derive $R(q)$. As in subsection 3.1, the thrust of the proof is to determine $R(q)$ for each of the five q ranges depicted in Fig. 2.

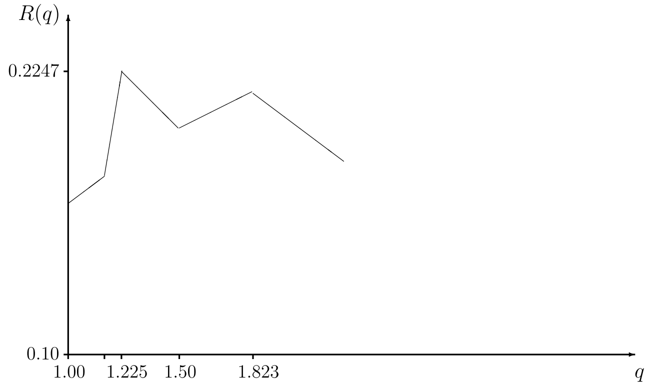

Fig. 2. Bound on $R(q)$ when $n \geqslant 4$.

Lemma 3. *For* $n = 4$, $R(q) \leqslant \frac{q}{3q+3}$, $q \in [1, \frac{1+\sqrt{37}}{6}]$, $R(q) \leqslant q - 1$, $q \in [\frac{1+\sqrt{37}}{6}, \sqrt{1.5}]$.

Proof. We observe first that for all ten nondominated optimal configurations listed earlier $n_{\mathrm{OPT}}(1) > 0$.

3-1. $n_{\mathrm{OPT}}(1) \geqslant 3$; then $C^*(P,q) \geqslant 3p_n = 3p_4$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{3p_n} = \frac{q}{3q+3}, \; q \geqslant 1. \tag{9}$$

The inequality $\frac{q}{3q+3} \leqslant q - 1$ when $q \in [\frac{1+\sqrt{37}}{6}, \sqrt{1.5}]$ completes the proof of 3-1.

3-2. $n_{\mathrm{OPT}}(1) = 1$; then $n_{\mathrm{OPT}}(2) = 3$, therefore $C^*(P,q) \geqslant 3qp_n = 3qp_4$, which, together with (2), yields

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{3qp_n} = \frac{1}{3q+3} \leqslant \frac{q}{3q+3}, \; q \geqslant 1.$$

The inequality $\frac{q}{3q+3} \leqslant q - 1$ when $q \in [\frac{1+\sqrt{37}}{6}, \sqrt{1.5}]$ completes the proof of 3-2.

3-3. $n_{\mathrm{OPT}}(1) = 2$ (which corresponds to the nondominated configurations $O(1,2|3,4)$, $O(1,3|2,4)$, $O(1,4|2,3)$, $O(2,3|1,4)$); then

$$C^*(P,q) \geqslant \max\{p_1 + p_4, p_2 + p_3\}. \quad \square \tag{10}$$

We want to show that

$$C^H(P,q) \leqslant \max\{q(p_1 + p_4), q(p_2 + p_3)\} \tag{11}$$

for all ten possible configurations of $C^H(P,q)$ when $n = 4$ because the combination of (11) and (10) will lead to

$$\mathrm{REL}(P,q) = \frac{C^H(P,q) - C^*(P,q)}{C^*(P,q)} \leqslant q - 1. \tag{12}$$

Inequality (11) is clearly true for configuration $H(1,4|2,3)$ (because $C^H(P,q) = \max\{p_1 + p_4, q(p_2 + p_3)\}$ for $H(1,4|2,3)$) and for configuration $H(2,3|1,4)$ (because $C^H(P,q) = \max\{p_2 + p_3, q(p_1 + p_4)\}$ for $H(2,3|1,4)$).

Consider the five configurations $H(1|2,3,4)$, $H(1,2,4|3)$, $H(1,3,4|2)$, $H(2,3,4|1)$, $H(1,2,3,4|-)$. For each one of them, let us consider the alternative configuration H' derived from H by assigning job J_4 to the other machine; denote this matching as $H \to H'$. Since H represents the MLPT solutions with the fourth jobs assigned according to the LPT rule, $C^H(P,q) \leqslant C^{H'}(P,q)$. We show next that inequality (11) holds for each one of the five $C^{H'}(P,q)$ values, therefore it also holds for each one of the corresponding $C^H(P,q)$ values since $C^H(P,q) \leqslant C^{H'}(P,q)$.

3-3-1. $H(1|2,3,4) \to H'(1,4|2,3)$; $C^{H'}(P,q) \leqslant p_1 + p_4$.
3-3-2. $H(1,2,4|3) \to H'(1,2|3,4)$; $C^{H'}(P,q) \leqslant q(p_3 + p_4)$.
3-3-3. $H(1,3,4|2) \to H'(1,3|2,4)$; $C^{H'}(P,q) \leqslant q(p_2 + p_4)$.
3-3-4. $H(2,3,4|1) \to H'(2,3|1,4)$; $C^{H'}(P,q) \leqslant q(p_1 + p_4)$.
3-3-5. $H(1,2,3,4|-) \to H'(1,2,3|4)$; $C^{H'}(P,q) \leqslant qp_4$.

It is easy to see from the above inequalities that (11) holds for all $C^{H'}(P,q)$ values listed above.

We now prove inequality (11) for the remaining three configurations $H(1,2|3,4)$, $H(1,3|2,4)$ and $H(1,2,3|4)$, respectively. Let the notation $H \leftarrow O$ denote that O is the optimal configuration for $n = 3$ that generated the H configuration for $n = 4$.

3-3-6. $H(1,2|3,4) \leftarrow O(1,2|3)$. The optimality of $O(1,2|3)$ implies that the $(1|2,3)$ configuration is no better than $O(1,2|3)$, or that

$$\max\{p_1 + p_2, qp_3\} \leqslant q(p_2 + p_3), \tag{13}$$

therefore $C^H(P,q) = \max\{p_1 + p_2, q(p_3 + p_4)\} \leqslant q(p_2 + p_3)$ because of (13).

3-3-7. $H(1,3|2,4) \leftarrow O(1,3|2)$. The optimality of $O(1,3|2)$ implies that the $(1|2,3)$ configuration is no better than $O(1,3|2)$, or that

$$\max\{p_1 + p_3, qp_2\} \leqslant q(p_2 + p_3), \tag{14}$$

therefore $C^H(P,q) = \max\{p_1 + p_3, q(p_2 + p_4)\} \leqslant q(p_2 + p_3)$ because of (14).

3-3-8. $H(1,2,3|4) \leftarrow O(1,2,3|-)$. The optimality of $O(1,2,3|-)$ implies that the $(1,2|3)$ configuration is no better than $O(1,2,3|-)$, or that

$$p_1 + p_2 + p_3 \leqslant qp_3, \tag{15}$$

therefore $C^H(P,q) = \max\{p_1 + p_2 + p_3, qp_4\} \leqslant q(p_2 + p_3)$ because of (15).

Consequently, inequality (11) holds for all H configurations and, together with (10), yields (12). The inequality $q - 1 \leqslant \frac{q}{3q+3}$ when $q \in [1, \frac{1+\sqrt{37}}{6}]$ completes the proof of 3-3.

Lemma 4. *For $n = 4$, $R(q) \leqslant \frac{1}{2q+2}$, $q \in [\sqrt{1.5}, 1.5]$, $R(q) \leqslant \frac{q}{3q+3}$, $q \in [1.5, \frac{1+\sqrt{7}}{2}]$.*

Proof

4-1. $n_{\text{OPT}}(1) \geqslant 3$; by Lemma 3 (case 3-1), inequality (9) holds. The inequality $\frac{q}{3q+3} \leqslant \frac{1}{2q+2}$ when $q \in [\sqrt{1.5}, 1.5]$ completes the proof of 4-1.

4-2. $n_{\text{OPT}}(2) \geqslant 2$; by Lemma 2 (case 2-2), inequality (8) holds. The inequality $\frac{1}{2q+2} \leqslant \frac{q}{3q+3}$ when $q \in [1.5, \frac{1+\sqrt{7}}{2}]$ completes the proof of 4-2. □

Lemmas 3 and 4 combined with Proposition 2 for $q \in [\frac{1+\sqrt{7}}{2}, \infty)$ lead to the following proposition which is stated next without proof (see also Fig. 2).

Proposition 4. *For $n = 4$, $\rho \leqslant \sqrt{1.5} - 1 = 0.2247$.*

3.3. The worst-case ratio bound of MLPT when $n = 5$

In this section we derive the ρ value for the MLPT heuristic when $n = 5$ by relying only on the properties of the optimal solution when $n = 5$ and on Propositions 1 and 2. As in the previous subsections, the thrust of the proof is to determine $R(q)$ for each of the four q ranges depicted in Fig. 3.

Lemma 5. *For $n = 5$, $R(q) \leqslant \frac{q}{3q+3}$, $q \in [1, 1.5]$.*

Proof. For any optimal solution, either $n_{\text{OPT}}(1) \geqslant 3$ or $n_{\text{OPT}}(2) \geqslant 3$, therefore $C^*(P,q) \geqslant 3p_n$, which according to Lemma 3 leads to inequality (9). □

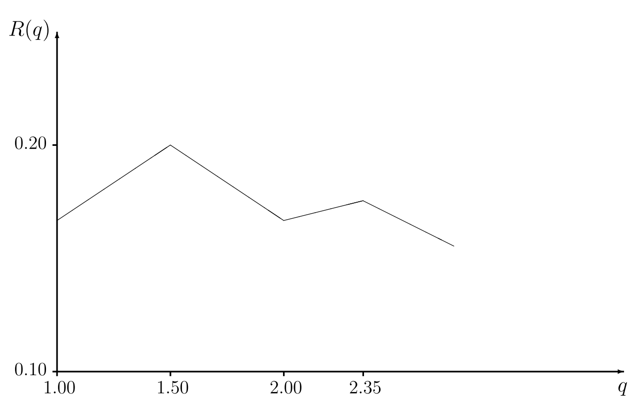

Fig. 3. Bound on $R(q)$ when $n \geqslant 5$.

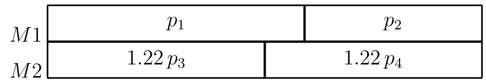

Fig. 4. The optimal schedule for the tightness example.

Lemma 6. *For $n = 5$, $R(q) \leqslant \frac{1}{2q+2}$, $q \in [1.5, 2]$, $R(q) \leqslant \frac{q}{4q+4}$, $q \in [2, \frac{3+\sqrt{41}}{4}]$.*

Proof. For any optimal solution, either $n_{\mathrm{OPT}}(1) \geqslant 4$ or $n_{\mathrm{OPT}}(2) \geqslant 2$, therefore $C^*(P, q) \geqslant \min\{4p_n, 2qp_n\}$, which, together with (2), leads to

$$\mathrm{REL}(P,q) \leqslant \frac{\frac{qp_n}{q+1}}{\min\{4p_n, 2qp_n\}} = \max\left\{\frac{\frac{qp_n}{q+1}}{4p_n}, \frac{\frac{qp_n}{q+1}}{2qp_n}\right\} = \max\left\{\frac{q}{4q+4}, \frac{1}{2q+2}\right\}.$$

Since $\frac{q}{4q+4} \leqslant \frac{1}{2q+2}$ when $q \in [1.5, 2]$ and $\frac{1}{2q+2} \leqslant \frac{q}{4q+4}$ when $q \in [2, \frac{3+\sqrt{41}}{4}]$, the lemma holds.

Lemmas 5 and 6 combined with Proposition 2 for $q \in [\frac{3+\sqrt{41}}{4}, \infty)$ lead to the following proposition which is stated next without proof (see also Fig. 3). □

Proposition 5. *For $n = 5$, $\rho \leqslant 0.2$.*

3.4. The overall nonparametric bound of MLPT

Theorem 1. *$\rho \leqslant 0.2247$, and this bound is tight.*

Proof. The upper bound follows from the combination of Propositions 3–5. In order to prove tightness consider a problem instance with $n = 4$, $q = \sqrt{1.5} = 1.2247$, $p_1 = 1$, $p_2 = p_3 = p_4 = \frac{q+1}{q+2} = 0.6899$. The MLPT and optimal solutions are depicted in Figs. 4 and 5, respectively. Since $C^*(P, q) = 1 + p_2 = \frac{2q+3}{q+2} = 1.6899$ and $C^H(P, q) = p_2 + p_3 + p_4 = \frac{3q+3}{q+2} = 2.0697$, $\rho = q - 1 = 0.2247$. □

4. Extensions

In this section we show that the MLPT heuristic can become more accurate (at the expense of additional computational time) by sequencing optimally 4 or even 5 jobs prior to the implementation of the LPT sequencing rule. The resulting ρ values can be obtained without any new derivations by observing that Propositions 1 and 2 were derived utilizing only the LPT properties of MLPT without relying on the actual number of jobs scheduled optimally by MLPT. Consequently, the proofs of Propositions 1 and 2 remain valid if we assume that either 4 or 5 jobs are initially sequenced optimally prior to the implementation of the LPT rule. This observation combined with the fact that the proof of Proposition 5 utilizes only Propositions 1, 2 and the properties of the optimal solution when $n = 5$ yield the following proposition (stated without proof) for a variant of the MLPT heuristic (MLPT_1) in which the first (longest) 4 jobs are sequenced optimally prior to the implementation of the LPT rule.

Proposition 6. *For $MLPT_1$, the overall nonparametric worst-case ratio bound ρ_1 satisfies $\rho_1 \leqslant 0.2$.*

Similarly, since the proof of Proposition 3 also utilizes only Propositions 1, 2 and the properties of the optimal solution when $n \geqslant 6$, we state (without proof) the following proposition for another variant of the MLPT heuristic (MLPT_2) in which the first (longest) 5 jobs are sequenced optimally before the LPT rule is implemented.

Proposition 7. *For $MLPT_2$, the overall nonparametric worst-case ratio bound ρ_2 satisfies $\rho_2 \leqslant 0.1667$.*

We have not established tightness for ρ_1, ρ_2 in Propositions 6, 7; consequently, it is theoretically possible that these bounds can be improved since they are derived using only the structure of MLPT and not any additional structure resulting from the optimal sequencing of additional jobs. However, any potential improvement is bounded by the inequalities $0.167 \leqslant \rho_1 \leqslant 0.2$ and $0.143 \leqslant \rho_2 \leqslant 0.167$, respectively, in view of the following two problem instances. Let $n = 5$, $q = 1$, $p_1 = p_2 = 3$, $p_3 = p_4 = p_5 = 2$. Then the MLPT_1 heuristic and optimal configurations are $H(1, 3, 5|2, 4)$ and

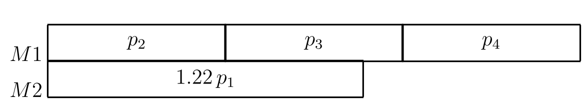

Fig. 5. The MLPT schedule for the tightness example.

$O(1,2|3,4,5)$, respectively. Since $C^*(P,q) = 6$ and $C^H(P,q) = 7$, $\rho_1 = 0.1667$. Let $n = 6$, $q = 1$, $p_1 = p_2 = 3$, $p_3 = p_4 = p_5 = p_6 = 2$. Then the MLPT$_2$ heuristic and optimal configurations are $H(1,2,6|3,4,5)$ and $O(1,3,4|2,5,6)$, respectively. Since $C^*(P,q) = 7$ and $C^H(P,q) = 8$, $\rho_2 = 0.143$.

5. Concluding remarks

We showed that the performance of the LPT heuristic for the $Q2\|C_{\max}$ problem can be improved by sequencing the longest three jobs optimally. Our results demonstrate the applicability of this approach (already implemented for identical parallel machine scheduling problems) to a uniform parallel machine environment. The ρ values decrease when more jobs are scheduled optimally, while there is no clear trend with respect to the corresponding q values. It should be noted that this approach cannot be extended indefinitely since the required computational effort for sequencing jobs optimally quickly becomes prohibitive due to the well-known NP-hardness of the $Q2\|C_{\max}$ problem.

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