



Short Communication

A unified analysis for the single-machine scheduling problem with controllable and non-controllable variable job processing times

Christos Koulamas*, Sushil Gupta, George J. Kyparisis

Department of Decision Sciences and Information Systems, Florida International University, Miami, FL 33199, United States

ARTICLE INFO

Article history:

Received 5 June 2009

Accepted 8 December 2009

Available online 14 December 2009

Keywords:

Scheduling

Single-machine

Variable processing times

ABSTRACT

We present a unified analysis for single-machine scheduling problems in which the actual job processing times are controlled by either a linear or a convex resource allocation function and also vary concurrently depending on either the job's position in the sequence and/or on the total processing time of the already processed jobs. We show that the problem is solvable in $O(n \log n)$ time by using a weight-matching approach when a convex resource allocation function is in effect. In the case of a linear resource allocation function, we show that the problem can be solved in $O(n^3)$ time by using an assignment formulation. Our approach generalizes the solution approach for the corresponding problems with controllable job processing times to incorporate the variability of the job processing times stemming from either the job's position in the sequence and/or the total processing time of the already processed jobs.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Single-machine scheduling problems with variable job processing times can be classified into: (i) problems with variable but controllable job processing times in which the scheduler's decision to allocate finite amounts of non-renewable resources to a job determines its processing time and (ii) problems with either learning and/or job deterioration considerations in which the job processing times vary beyond the control of the scheduler because of either the job's position in the sequence and/or the total processing time of the already processed jobs. Both categories have been studied independently and extensively in the scheduling literature. However, to the best of our knowledge, there is no published research that combines controllable and non-controllable job processing times in a single model.

The objective of this paper is to provide a unified analysis for single-machine scheduling problems by generalizing the solution approach for scheduling problems with controllable job processing times to incorporate the variability of the job processing times stemming from either the job's position in the sequence and/or from the total processing time of the already processed jobs.

The rest of the paper is organized as follows. In Section 2 a brief review of the existing models with variable processing times is presented. Our unified analysis for single-machine scheduling problems with variable (both controllable and non-controllable) processing times is presented in Section 3 along with some

comments about its applicability to other problems and its limitations.

2. Review of existing models

In this paper, we consider a standard single-machine scheduling problem with a batch of n jobs available for processing at time zero on a continuously available non-preemptive machine. Let p_j denote the nominal processing time of job J_j , $j = 1, \dots, n$; also, let $J_{[j]}$, $p_{[j]}$ denote the job occupying the j position in the sequence and its nominal processing time, respectively.

Previous research on single-machine scheduling with non-controllable variable job processing times has dealt with problems in which the actual processing time of a job depends on either the job's position in the sequence and/or on the total processing time of the already processed jobs.

Position-dependent job processing times have been utilized in the presence of learning. Biskup (1999) was the first to incorporate learning in scheduling problems; Biskup (1999) incorporated the learning phenomenon by defining the actual processing time of job J_j when scheduled in position r in the sequence, $p_{j|r}$, as

$$p_{j|r} = p_j r^\alpha, \quad (1)$$

where $\alpha < 0$ is the applicable learning rate.

The case of actual job processing times dependent upon the sum of the processing times of the already processed jobs has been used to model job deterioration due to waiting. Browne and Yechiali (1990) were among the first to introduce this type of job processing times. If we assume a uniform job deterioration rate

* Corresponding author. Tel.: +1 305 348 3309; fax: +1 305 348 4126.
E-mail address: koulamas@fiu.edu (C. Koulamas).

$\gamma \geq 0$ for all jobs, then the actual processing time of job J_j when scheduled in position r in the sequence, $p_{[r]}$ is given as

$$p_{[r]} = p_j + \gamma \sum_{i=1}^{r-1} p_{[i]}. \quad (2)$$

It should be pointed out that some researchers (e.g. Kuo and Yang, 2006) use variants of Eq. (2) to model sum-of-processing-time-dependent learning. A recent survey of position-dependent and sum-of-processing-times based learning models is presented by Biskup (2008).

There is also significant literature on scheduling with controllable variable job processing times. The common element of these models is that the actual job processing times are controlled through the allocation of a finite amount of a non-renewable resource. Vickson (1980) was among the first to consider such a model assuming that the actual job processing times are linear functions of the allocated amount of resource. Specifically, under the simplifying assumption that one unit x_j of the resource allocated to job J_j reduces its actual processing time p_{aj} by one time unit, the function $p_{aj}(x_j)$ can be defined as

$$p_{aj}(x_j) = p_j - x_j, \quad 0 \leq x_j \leq G_j, \quad j = 1, \dots, n, \quad (3)$$

where p_j is the nominal “uncompressed” processing time of job J_j and G_j is the maximum amount of time by which J_j can be compressed. It is of interest to notice that in the above special case of the one-for-one tradeoff between resource and time, x_j can also be viewed as the actual amount of time by which J_j is compressed.

An alternative controllable job processing times model utilizes a convex $p_{aj}(x_j)$ function given as

$$p_{aj}(x_j) = \left(\frac{p_j}{x_j}\right)^k, \quad j = 1, \dots, n, \quad (4)$$

where $k > 0$ is a positive constant. Monma et al. (1990) were among the first to propose (4) in the context of a graph theory problem.

The presence of controllable job processing times necessitates the consideration of composite (total cost) objective functions comprised of both a scheduling criterion and the cost of the allocated resource. In a single-machine environment, the total cost function can be written as

$$TC(x_{[j]}) = \sum_{j=1}^n [x_{[j]}c_{[j]} + p_{a[j]}(x_{[j]})w_{[j]}], \quad (5)$$

where $c_{[j]}$ denotes the cost of allocating one unit of resource $x_{[j]}$ to job $J_{[j]}$; $p_{a[j]}(x_{[j]})$ is given by either (3) or (4) and $w_{[j]}$ denotes the contribution of job $J_{[j]}$ to the objective function; $w_{[j]}$ is also known as the positional weight of job $J_{[j]}$.

There is a substantial body of literature on scheduling problems employing the objective function (5). A recent survey of this literature is presented by Shabtay and Steiner (2007a). The majority of this literature focuses on scheduling problems in which the job positional weights $w_{[j]}$ are independent of the actual job processing times $p_{a[j]}(x_{[j]})$. These problems can be categorized in the following two general categories.

In the first category, the scheduling criterion used in the objective function (5) is some function of the job completion times C_j , $j = 1, \dots, n$ which is either a variant or a special case of Bagchi's (1989) bi-criteria objective function

$$f(C_{j=1, \dots, n}) = \delta \sum_{j=1}^n C_j + (1 - \delta) \sum_{i=1}^n \sum_{j=1}^n |C_i - C_j|, \quad 0 \leq \delta \leq 1, \quad (6)$$

which can be written as $f(C_{j=1, \dots, n}) = \sum_{j=1}^n w_{[j]} p_{a[j]}(x_{[j]})$ with the positional weights $w_{[j]}$ given as

$$w_{[j]} = (2\delta - 1)(n + 1) + j[2 - 3\delta + n(1 - \delta)] - j^2(1 - \delta), \quad j = 1, \dots, n. \quad (7)$$

In the second category, the objective is to determine the job due dates in order to minimize a variant of the objective function (5) containing a due date-related objective. This line of research was motivated by the work of Panwalkar et al. (1982) who considered the single-machine common due date assignment problem with total earliness and tardiness penalties (and constant job processing times). It was shown by Panwalkar et al. (1982) that the resulting job positional weights are independent of the actual job processing times like the ones given by (7).

Shabtay and Steiner (2008) considered the single-machine due date assignment problem with controllable job processing times (given by either (1) or (2)) and three different due date assignment methods (the common due date, the equal slack and the unrestricted due date assignment methods). It was observed that the independence of the job positional weights from the actual job processing times is retained in the presence of controllable job processing times. This observation facilitated the solution of the problem (for any due date assignment method) in $O(n \log n)$ time (by using the weight-matching approach of Hardy et al. (1967)) when the actual job processing times are given by (1) and in $O(n^3)$ time (by using assignment formulations) when the actual job processing times are given by (2).

We close the review of existing models by mentioning that Shabtay and Steiner (2007b) considered similar due date assignment problems as in Shabtay and Steiner (2008) with the weighted number of tardy jobs objective. Shabtay and Steiner (2007b) exploited the fact that the two sets of problems exhibit similar structure to derive polynomial-time algorithms for the weighted number of tardy jobs case as well.

3. A unified analysis for single-machine scheduling

In this section, we show that the solution techniques used to solve single-machine scheduling problems with controllable job processing times can be extended to incorporate the concurrent variability of the actual job processing times stemming from either learning and/or job deterioration. Specifically, we propose a unified analysis to minimize the objective function (5) when the processing times are controlled according to either (3) or (4) while they also vary according to either (1) and/or (2).

3.1. The convex processing times case

We first consider the case in which the actual job processing times are given by the convex function (4); we initially assume that neither Eq. (1) nor Eq. (2) are in effect. In that case, the analysis by Shabtay and Steiner (2008) shows that the substitution of (4) into (5) yields $TC(x_{[j]}) = \sum_{j=1}^n \left\{ x_{[j]} c_{[j]} + \left(\frac{p_{[j]}}{x_{[j]}}\right)^k w_{[j]} \right\}$ which at the optimal resource allocation level can be written as

$$TC(x_{[j]}) = \tilde{k} \times \sum_{j=1}^n (w_{[j]})^{\frac{1}{k+1}} \tilde{p}_{[j]}, \quad (8)$$

where $\tilde{k} = k^{\frac{k}{k+1}} + k^{\frac{1}{k+1}}$ and $\tilde{p}_{[j]} = (p_{[j]} c_{[j]})^{\frac{k}{k+1}}$, $j = 1, \dots, n$. The structure of the objective function (8) facilitates the solution of more complex problems (with no additional computational effort) in which the actual job processing time p_{aj} also depends on the position of job J_j in the sequence (according to either Eq. (1) and/or Eq. (2)) by simply adjusting the positional weights $w_{[j]}$ accordingly and then using the weight-matching approach of Hardy et al. (1967).

In the case of learning with Eq. (1) in effect, the completion time-based weights $w_{[j]}$ given by (7) should be replaced by $j^{2k} w_{[j]}$ in (8). In the case of job deterioration when Eq. (2) are in effect, the substitution of Eq. (2) into (6) leads after some algebra to expressing (6) as $f(C_{j=1, \dots, n}) = \sum_{j=1}^n v_{[j]} p_{a[j]}(x_{[j]})$ where

Table 1
The positional weights under learning and/or job deterioration effects.

Actual job processing time	Learning	Deterioration	Learning/deterioration
$p_{aj}(x_j) = \left(\frac{p_j}{x_j}\right)^k$	$(j^{zk} w_{jj})^{\frac{1}{k+1}}$	$(v_{jj})^{\frac{1}{k+1}}$	$(j^{zk} v_{jj})^{\frac{1}{k+1}}$
$p_{aj}(x_j) = p_j - x_j$	$j^{zk} w_{jj}$	v_{jj}	$j^{zk} v_{jj}$

$$v_{jj} = \delta(n - j + 1) \left(1 + \gamma \frac{n - j}{2}\right) + (1 - \delta) \left[(j - 1)(n - j + 1) + \gamma \sum_{r=j+1}^n (r - 1)(n - r + 1) \right]. \quad (9)$$

Consequently, the completion time-based weights w_{jj} given by (7) should be replaced by the ones given by (9) in the objective function (8).

The weight-matching approach is applicable even when all functions (1), (2), and (4) are concurrently in effect. This extension applies to practical situations in which the actual job processing times are affected by both the position of the job in the sequence and the total processing time of the already processed jobs. In that case, the completion time-based weights w_{jj} given by (7) should be replaced by $j^{zk} v_{jj}$ in the objective function (8). We summarize our findings in the first row of Table 1.

Our analysis is not limited to the case in which the w_{jj} values are given by (7); instead, it can be also applied to the due date assignment problems considered by Shabtay and Steiner (2008) when the convex processing times given by Eq. (2) are in effect.

3.2. The linear processing times case

We now consider the case in which the actual job processing times p_{aj} are given by the linear functions (3). We initially assume that neither Eq. (1) nor Eq. (2) are in effect. In that case, it is easy to observe that the contribution of job J_{jj} to the objective function (5) is $x_{jj}c_{jj} + w_{jj}(p_{jj} - x_{jj})$. In order to minimize (5), it is clear that if $c_{jj} \geq w_{jj}$, then $x_{jj} = 0$ and if $c_{jj} \leq w_{jj}$, then $x_{jj} = G_{jj}$. This observation is in agreement with the well-known result from prior research in this area (e.g. Vickson, 1980) that no job is partially compressed in an optimal solution and leads to an assignment formulation to minimize (5). Define x_{ij} as a binary variable equal to one if job J_i is assigned to position j and equal to zero otherwise. The cost of assigning J_i to position j , θ_{ij} is given as

$$\theta_{ij} = \begin{cases} p_i w_j, & c_i > w_j, \\ (p_i - G_i) w_j + c_i G_i, & c_i \leq w_j. \end{cases}$$

Then, the assignment formulation (A1) with the objective of minimizing $\sum_{i=1}^n \sum_{j=1}^n \theta_{ij} x_{ij}$ subject to $\sum_{i=1}^n x_{ij} = 1, j = 1, \dots, n, \sum_{j=1}^n x_{ij} = 1, i = 1, \dots, n$, and $x_{ij} \in \{0, 1\}, i, j = 1, \dots, n$ can be used to determine a sequence which minimizes the objective function (5) in $O(n^3)$ time.

The assignment formulation (A1) was used by Wang and Xia (2007). Shabtay and Steiner (2008) developed similar assignment formulations to solve single-machine due date assignment problems with linear job processing times (given by (3)) and with positional weights w_{jj} corresponding to the common due date, the equal slack due date and the unrestricted due date assignment methods, respectively. Panwalkar and Rajagopalan (1992), Cheng et al. (1996), Biskup and Jahnke (2001), Ng et al. (2003) and Alidaee and Ahmadian (1993) studied special cases of the Shabtay and Steiner (2008) models primarily for the common due date assignment method. Biskup (1999) and Mosheiov (2001) generalized Panwalkar et al.'s (1982) constant processing times due date assignment model to a learning environment in which the functions (1) are

in effect (with no resource allocation decisions) and solved the problem in $O(n^3)$ time by using assignment formulations.

As in the convex processing times case, the assignment formulation (A1) can be generalized to incorporate the concurrent non-controllable variability of the actual job processing times given by either Eq. (1) and/or by Eq. (2) by adjusting appropriately the job positional weights w_{jj} . In the case of the completion time-based positional weights given by (7), these adjustments are summarized in the second row of Table 1. As in the convex processing times case, our analysis is not limited to the case in which the job positional weights w_{jj} are given by (7); instead, it can be also implemented using the job positional weights corresponding to the three due date assignment models of Shabtay and Steiner (2008).

In principle, any variant of the due date assignment model which retains the independence of the job positional weights from the actual job processing times can be generalized (by implementing our unified analysis) to incorporate the non-controllable variability of the job processing times given by either Eq. (1) and/or Eq. (2). As an example of models amenable to our analysis, we mention the model of Liman et al. (1996) who generalized Panwalkar et al.'s (1982) common due date problem (with constant job processing times) to a common due window and also the model of Liman et al. (1997) who generalized Panwalkar and Rajagopalan (1992) common due date problem (with linear controllable job processing times) to a common due window problem.

A common characteristic of the class of problems exhibiting this type of structure is that the contribution of each job to the objective function depends only on the number of the already sequenced jobs or on the number of the remaining jobs to be scheduled but not on the identity of these jobs. This type of structure is usually lost when job weights are introduced; for example, the non-weighted total job completion time $\sum_{j=1}^n C_{jj}$ can be expressed as $\sum_{j=1}^n (n - j + 1) p_{jj}$ while this is not the case for its weighted counterpart $\sum_{j=1}^n b_{jj} C_{jj}$ where b_j denotes the weight (importance) of job J_j . A similar situation occurs when job-specific deterioration rates $\gamma_j, j = 1, \dots, n$ are in effect. In these cases, the positional weights w_{jj}, v_{jj} can no longer be expressed as in (7) and (9), respectively, and our unified solution approach is no longer applicable.

Acknowledgements

We would like to thank the referees for their insightful comments which helped us improve an earlier version of this paper.

References

Alidaee, B., Ahmadian, A., 1993. Two parallel machine sequencing problems involving controllable job processing times. *European Journal of Operational Research* 70, 335–341.

Bagchi, U.B., 1989. Simultaneous minimization of mean and variation of flow-time and waiting time in single machine systems. *Operations Research* 37, 118–125.

Biskup, D., 1999. Single-machine scheduling with learning considerations. *European Journal of Operational Research* 115, 173–178.

Biskup, D., 2008. A state-of-the-art review of scheduling with learning effects. *European Journal of Operational Research* 188, 315–329.

Biskup, D., Jahnke, D., 2001. Common due date assignment for scheduling on a single machine with jointly reducible processing times. *International Journal of Production Economics* 69, 317–322.

Browne, S., Yechiali, U., 1990. Scheduling deteriorating jobs on a single processor. *Operations Research* 38, 495–498.

Cheng, T.C.E., Ogaz, C., Qi, X.D., 1996. Due-date assignment and single machine scheduling with compressible processing times. *International Journal of Production Economics* 43, 29–35.

Hardy, G.H., Littlewood, J.E., Polya, G., 1967. *Inequalities*. Cambridge University Press, London.

Kuo, W.-H., Yang, D.-L., 2006. Minimizing the total completion time in a single machine scheduling problem with a time-dependent learning effect. *European Journal of Operational Research* 174, 1184–1190.

- Liman, S.D., Panwalkar, S.S., Thongmee, S., 1996. Determination of common due window location in a single machine scheduling problem. *European Journal of Operational Research* 93, 68–74.
- Liman, S.D., Panwalkar, S.S., Thongmee, S., 1997. A single machine scheduling problem with common due window and controllable processing times. *Annals of Operations Research* 70, 145–154.
- Monma, C.L., Schrijver, A., Todd, M.J., Wei, V.K., 1990. Convex resource allocation problems on directed acyclic graphs: Duality, complexity, special cases and extensions. *Mathematics of Operations Research* 15, 736–748.
- Mosheiov, G., 2001. Scheduling problems with a learning effect. *European Journal of Operational Research* 132, 687–693.
- Ng, C.T.D., Cheng, T.C.E., Kovalyov, M.Y., Lam, S.S., 2003. Single machine scheduling with a variable common due date and resource-dependent processing times. *Computers & Operations Research* 30, 1173–1185.
- Panwalkar, S.S., Rajagopalan, R., 1992. Single-machine sequencing with controllable processing times. *European Journal of Operational Research* 59, 298–302.
- Panwalkar, S.S., Smith, M.L., Seidman, A., 1982. Common due date assignment to minimize total penalty for the one machine scheduling problem. *Operations Research* 30, 391–399.
- Shabtay, D., Steiner, G., 2007a. A survey of scheduling with controllable processing times. *Discrete Applied Mathematics* 155, 1643–1666.
- Shabtay, D., Steiner, G., 2007b. Optimal due date assignment and resource allocation to minimize the weighted number of tardy jobs on a single machine. *Manufacturing & Service Operations Management* 9, 332–350.
- Shabtay, D., Steiner, G., 2008. The single-machine earliness–tardiness scheduling problem with due date assignment and resource-dependent processing times. *Annals of Operations Research* 159, 25–40.
- Vickson, R.G., 1980. Two single-machine sequencing problems involving controllable processing times. *AIIE Transactions* 12, 258–262.
- Wang, J.B., Xia, Z.Q., 2007. Single machine scheduling problems with controllable processing times and total absolute differences penalties. *European Journal of Operational Research* 177, 638–645.